

Deforming an ϵ -Close to Hyperbolic Metric to a Hyperbolic Metric

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Abstract

We show how to deform a metric of the form $g = g_r + dr^2$ to a hyperbolic metric $\mathcal{H}g$, for r less than some fixed λ . We study to extent the *hyperbolic forced metric* $\mathcal{H}g$ is *close to being hyperbolic*, if we assume g to be close to hyperbolic. We also deal with a one-parameter version of this problem.

The results in this paper are used in the problem of smoothing Charney-Davis strict hyperbolizations [1], [2].

Section 0. Introduction.

First we introduce some notation. Let (M^n, g) be a complete Riemannian manifold *with center* $o \in M$, that is, the exponential map $\exp_o : T_o M \rightarrow M$ is a diffeomorphism. In this case we can write the metric g on $M - \{o\} = \mathbb{S}^{n-1} \times \mathbb{R}^+$ as $g = g_r + dr^2$, where r is the distance to o . The ball of radius r in M , centered at o , will be denoted by $B_r = \mathbb{B}_r(M)$.

Let M have center o and metric $g = g_r + dr^2$. In general, the metric g is not a warped metric (i.e. $g_r = f(r)g'$ for some g' not depending on r). In [4] we gave a geometric process called *warp forcing* whose input is a metric of the form $g_r + dr^2$ on M , and output is the *warped forced* metric $\mathcal{W}_{r_0}g$ on M (see section 3 for more details). Here $r_0 > 0$ is pre-fixed number. The metric $\mathcal{W}_{r_0}g$ has the following property

$$(0.1) \quad \mathcal{W}_{r_0}g = \begin{cases} \sinh^2(r)\hat{g}_{r_0} + dr^2 & \text{on } B_{r_0} \\ g & \text{outside } B_{r_0 + \frac{1}{2}} \end{cases}$$

where

$$(0.2) \quad \hat{g}_{r_0} = \left(\frac{1}{\sinh^2(r_0)} \right) g_{r_0}$$

Hence warp forcing changes the metric only on $B_{r_0 + \frac{1}{2}}$, making it a warped metric inside B_{r_0} . The metric g does not change outside $B_{r_0 + \frac{1}{2}}$.

Remark. We can think of the metric g_{r_0} in as being obtained from $g = g_r + dr^2$ by “cutting” g along the sphere of radius r_0 , so we call g_{r_0} the *warped spherical cut of g at r_0* . Also we call the metric \hat{g}_{r_0} in (0.2) the *(unwarped) spherical cut of g at r_0* . In the particular case that $g = g_r + dr^2$ is a warped-by- \sinh metric we have $g_r = \sinh^2(r)g'$ for some fixed g' independent of r . In this case the warped spherical cut of $g = \sinh^2(r)g' + dr^2$ at r_0 is $\sinh^2(r_0)g'$, and the spherical cut at r_0 is $\hat{g}_{r_0} = g'$. Hence the terms “warped” and “unwarped” (usually we will omit the term “unwarped”).

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On the other hand in [3] we introduced another geometric construction, the *two variable warping trick* (see section 2 for more details). Its input is a warped metric $g = \sinh^2(r)h + dr^2$ on $\mathbb{S}^{n-1} \times \mathbb{R}^+$, and its output is the metric $\mathcal{T}_{a,d}g$, where $a, d > 0$ are two parameters. The metric $\mathcal{T}_{a,d}g$ has the following property.

$$(0.3) \quad \mathcal{T}_{a,d}g = \begin{cases} \sinh^2(r)\sigma_{\mathbb{S}^{n-1}} + dr^2 & \text{on } B_a \\ g & \text{outside } B_{a+\frac{d}{2}} \end{cases}$$

Here $\sigma_{\mathbb{S}^{n-1}}$ is the canonical round metric on the sphere. Hence, the two variable warping trick changes a warped metric h inside the ball $B_{a+\frac{d}{2}}$ making it (radially) hyperbolic on the smaller ball B_a . The warped metric h does not change outside $B_{a+\frac{d}{2}}$.

Remark 0.4 To be able to define $\mathcal{T}_{a,d}g$ the metric h does not need to be a warped metric everywhere. It only needs to be a warped metric in the ball $B_{a+\frac{d}{2}}$. Also to define $\mathcal{W}_{r_0}g$, the metric g needs to be defined only outside the interior of B_{r_0} .

In this paper we consider the composition of these two geometric constructions. Given $g = g_r + dr^2$, $d > 0$, r_0 with $r_0 > d$, we define the metric $\mathcal{H}_{r_0,d}(g)$ in the following way. First warp-force the metric g , i.e take $\mathcal{W}_{r_0}g$. Since $\mathcal{W}_{r_0}g$ is a warped metric on B_{r_0} (see (0.1)) we can use the two variable warping trick (see remark 0.4) and define

$$(0.5) \quad \mathcal{H}_{r_0,d}g = \mathcal{T}_{(r_0-d),d}(\mathcal{W}_{r_0}g)$$

We call by *hyperbolic forcing (with cut at r_0)* the process given by the correspondance $g \mapsto \mathcal{H}_{r_0,d}g$. We have

$$(0.6) \quad \mathcal{H}_{r_0,d}g = \begin{cases} \sinh^2(r)\sigma_{\mathbb{S}^{n-1}} + dr^2 & \text{on } B_{r_0-d} \\ \text{warped metric} & \text{on } B_{r_0} - B_{r_0-\frac{d}{2}} \\ g \text{ (no change)} & \text{outside } B_{r_0+\frac{1}{2}} \end{cases}$$

Hence the metric $\mathcal{H}_{r_0,d}g$ is hyperbolic inside B_{r_0-d} . In section 4 we give some properties of this metric.

In this paper we prove that if g is in some sense “close to being hyperbolic” then the hyperbolic forced metric $\mathcal{H}_{r_0,d}$ is also, to some extent, close to hyperbolic. In the next paragraph we explain what we mean by a metric being close to hyperbolic (for more details see section 1). Essentially, our definition uses charts.

Let $\mathbb{B} \subset \mathbb{R}^{n-1}$ be the unit $(n-1)$ -ball, with the flat metric $\sigma_{\mathbb{R}^{n-1}}$. Write $I_\xi = (-(1+\xi), 1+\xi)$, $\xi > 0$. Our basic models are $\mathbb{T}_\xi = \mathbb{B} \times I_\xi$, with hyperbolic metric $\sigma = e^{2t}\sigma_{\mathbb{R}^{n-1}} + dt^2$. The number ξ is called the *excess* of \mathbb{T}_ξ . (The reason for introducing ξ is because warp forcing and hyperbolic forcing reduce the excess of the charts). Let (M, g) be a Riemannian manifold and $S \subset M$. We say that S is *ϵ -close to hyperbolic* if there is $\xi > 0$ such that for every $p \in S$ there is an *ϵ -close to hyperbolic chart with center p* , that is, there is a chart $\phi : \mathbb{T}_\xi \rightarrow M$, $\phi(0,0) = p$, such that $|\phi^*g - \sigma|_{C^2} < \epsilon$. The number ξ is called the *excess* of the charts (which is fixed).

If M^n has center o we say that $S \subset M$ is *radially ϵ -close to hyperbolic (with respect to o)* if, in addition, the charts ϕ respect the product structure of \mathbb{T} and $M - o = \mathbb{S}^{n-1} \times \mathbb{R}^+$, that is $\phi(.,t) = (\phi_1(.), t+a)$, for some constant a depending only on the ϕ (see Section 2 for details). Here the “radial” directions are $(-(1+\xi), 1+\xi)$ and \mathbb{R}^+ in \mathbb{T} and $M - o$, respectively.

Remark 0.7. The definition of radially ϵ -close to hyperbolic metrics is well suited to studying metrics for t large, but for small t this definition is not useful because of: (1) the need for some space to fit the charts, and (2) the form of our specific fixed model \mathbb{T} . An undesired consequence is that punctured hyperbolic space $\mathbb{H}^n - \{o\} = \mathbb{S}^{n-1} \times \mathbb{R}^+$ (with warp metric $\sinh^2(t)\sigma_{\mathbb{S}^{n-1}} + dt^2$) is not radially ϵ -close to hyperbolic for t small. In fact there is $\mathfrak{a} = \mathfrak{a}(n, \epsilon)$ such that hyperbolic n -space is ϵ -close to hyperbolic for $t > \mathfrak{a}$ (and not for all $t \leq \mathfrak{a}$). We prove this in 3.9 of [3].

As mentioned above our first result says that if the metric $g = g_r + dr^2$ is radially ϵ -close to hyperbolic then the hyperbolic forced metric $\mathcal{H}_{r_0, d}$ is radially η -close to hyperbolic, where η depends on ϵ , r_0 and d . Recall that $\mathcal{H}_{r_0, d}$ is hyperbolic on B_{r_0-d} (see (0.3), (0.5)). This motivates us to deal with a natural and useful class of metrics. These are metrics on \mathbb{R}^n (or on a manifold with center) that are already hyperbolic on the ball $B_a = B_a(0)$ of radius a centered at 0, and are radially ϵ -close to hyperbolic outside $B_{a'}$ (here a' is slightly less than a). Here is the detailed definition. Let M^n have center o and let $B_a = B_a(o)$ be the ball on M of radius a centered at o . We say that a metric h on M is (B_a, ϵ) -close to hyperbolic, with charts of excess ξ , if

- (1) On $B_a - \{o\} = \mathbb{S}^{n-1} \times (0, a)$ we have $h = \sinh^2(t)\sigma_{\mathbb{S}^{n-1}} + dt^2$. Hence h is hyperbolic on B_a .
- (2) the metric h is radially ϵ -close to hyperbolic outside $B_{a-1-\xi}$, with charts of excess ξ .

Remarks.

1. We have dropped the word “radially” to simplify the notation. But it does appear in condition (2), where “radially” refers to the center of B_a .
2. We will always assume $a > \mathfrak{a} + 1$, where \mathfrak{a} is as in 0.6. Therefore conditions (1), (2) and remark 0.6 imply a stronger version of (2):

(2') the metric h is radially ϵ -close to hyperbolic outside B_a , with charts of excess ξ .

This is the reason why we demanded radius $a - 1 - \xi$ in (2), instead of just a .

Metrics that are (B_a, ϵ) -close to hyperbolic are very useful, and are key objects in [2]. See also [3], [5]. Our next result answers the following question:

To what extent is the hyperbolically forced metric $h = \mathcal{H}_{r_0, d} g$ close to hyperbolic, when g is close to hyperbolic?

Before we state our first result we need one more definition. A family g_t of metrics on the sphere is c -bounded if each metric g_t (and its derivatives up to order 2) are bounded by c (see section 1 for more details). We are ready to state our first result. Recall that \hat{g}_{r_0} is the spherical cut of g at r_0 (see 0.2). In the next theorem we assume $d \geq 4(1 + \xi)$.

Theorem 1. *Let M^n have center o and metric $g = g_r + dr^2$. Assume the spherical cut \hat{g}_{r_0} is c -bounded. If the metric g is radially ϵ -close to hyperbolic outside $B_{r_0-(1+\xi)}$ with charts of excess $\xi > 1$, then the metric $\mathcal{H}_{r_0, d} g$ is (B_{r_0-d}, η) -close to hyperbolic with charts of excess $\xi - 1$, provided*

$$\eta \geq C_1 \left(\frac{1}{d} + e^{-(r_0-d)} \right) + C_2 \epsilon$$

Here C_1 is a constant depending only on n , ξ , c , and C_2 depends only on ξ .

Remarks 0.8.

1. An important point here is that by taking r_0 and d large we can make $\mathcal{H}_{r_0,d} g$ $2C_2\epsilon$ -close to hyperbolic. How large we have to take d and r_0 depends on c , which is a C^2 bound for the the \hat{g}_{r_0} , the spherical cut of g at r_0 (see 0.2).
2. We give a formula for $C_1 = C_1(c, n, \xi)$ at the end of Section 4. We can take $C_2 = e^{16+6\xi}$.
3. Note that the excess of the charts decreases by 1.

Next we give a one-parameter version of hyperbolic forcing with cut at r_0 , with the variable $r_0 \rightarrow \infty$ (we change notation and use λ instead of r_0 to express the fact that this number now varies). This family version of hyperbolic forcing is an important ingredient in [2], where it is used to smooth the singularities of a Charney-Davis strict hyperbolization [1].

Let M^n have center o . As before write $M - \{o\} = \mathbb{S}^{n-1} \times \mathbb{R}^+$. Recall that to apply hyperbolic forcing, with cut at r_0 , to a metric $g = g_r + dr^2$, the metric g needs to be defined only for $r > r'$, where $r' < r_0$ (see remark 0.4).

Fix ξ . We now consider families of metrics $\{g_\lambda\}_{\lambda > \lambda_0}$ of the form $g_\lambda = (g_\lambda)_r + dr^2$. Here $\lambda_0 > 1 + \xi$. We will assume that $(g_\lambda)_r$ defined for (at least) $r > \lambda - (1 + \xi)$. We call such a family a \odot -family of metrics. We now apply hyperbolic forcing to each g_λ with cut at $r = \lambda$, that is we consider $\mathcal{H}_{\lambda,d} g_\lambda$. To simplify write $\mathcal{H}_d g_\lambda = \mathcal{H}_{\lambda,d} g_\lambda$. Note that to obtain $\mathcal{H}_d g_\lambda$ from g_λ we are cutting at λ , that is we are cutting g_λ “along” the sphere of radius λ .

We want to give a one-parameter version of Theorem 1, that is a version for a \odot -family $\{g_\lambda\}$. A key fact in Theorem 1 is that the constant C depends on the bound c of the spherical cut \hat{g}_{r_0} . But now for each g_λ we get a bound c_λ . Hence for each g_λ we get a constant C_λ . And if it happens that $c_\lambda \rightarrow \infty$, we also get $C_\lambda \rightarrow \infty$. Therefore there is no C as in Theorem 1 that would work for every g_λ . This problem motivates the following definition.

We say that the \odot -family $\{g_\lambda\}$ has *cut limit at b* if there is a C^2 metric $\hat{g}_{\infty+b}$ on \mathbb{S}^n such that

$$\widehat{(g_\lambda)}_{\lambda+b} \xrightarrow{C^2} \hat{g}_{\infty+b}$$

Recall that the “hat” notation means “spherical cut” (see 0.1). One more definition. We say that an \odot family $\{g_\lambda\}$ is radially ϵ -close to hyperbolic, with charts of excess ξ , if each g_λ is radially ϵ -close to hyperbolic outside $B_{\lambda-(1+\xi)}$, with charts of excess ξ . The following is a corollary of Theorem 1, and is proved in section 5.

Theorem 2. *Let M have center o , and $\{g_\lambda\}$ an \odot -family on M . Assume that $\{g_\lambda\}$ has cut limits at $b = 0$, i.e. $\widehat{(g_\lambda)}_\lambda \rightarrow \hat{g}_\infty$. If $\{g_\lambda\}$ is radially ϵ -close to hyperbolic, with charts of excess $\xi > 1$, then $\mathcal{H}_d g_\lambda$ is $(B_{\lambda-d}, \eta)$ -close to hyperbolic, with charts of excess $\xi - 1$, where*

$$\eta \geq C_1 \left(e^{-(\lambda-d)} + \frac{1}{d} \right) + C_2 \epsilon$$

Here $C_2 = C_2(\xi)$ and $C_1 = C_1(c, n, \xi)$ (for some c) are as in Theorem 1.

Remark. An important point here is that the constant $C_1 = C_1(c, n, \xi)$ works for all $\mathcal{H}_d g_\lambda$. Note that nothing is said about the constant c , only that it exist. In fact c is such that $\{\widehat{(g_\lambda)}_\lambda\}_\lambda$ is c -bounded. Such a c always exist (see Lemma 5.5).

Corollary. *Let M have center o , $\{g_\lambda\}$ an \odot -family on M , and $\epsilon' > 0$. Assume that $\{g_\lambda\}$ has cut limits at $b = 0$. If $\{g_\lambda\}$ is radially ϵ -close to hyperbolic, with charts of excess $\xi > 1$, then $\mathcal{H}_d g_\lambda$ is $(B_{\lambda-d}, \epsilon' + C_2\epsilon)$ -close to hyperbolic, with charts of excess $\xi - 1$, provided*

$$(i) \quad \lambda - d > \ln\left(\frac{2C_1}{\epsilon'}\right)$$

$$(ii) \quad d \geq \frac{2C_1}{\epsilon'}.$$

Here C_1 and C_2 are as in Theorem 2.

Note that we can take ϵ' as small as we want hence $\epsilon' + C_2\epsilon$ as close as $C_2\epsilon$ as we desire, provided we choose d and λ sufficiently large. How large depending on ϵ' and c .

To deduce Corollary from Theorem 2 just note that (i) and (ii) in Corollary imply $C_1 e^{-(\lambda-d)} < \epsilon'/2$ and $C_1 \frac{1}{d} < \epsilon'/2$.

In section 1 we introduce some notation and define ϵ -close to hyperbolic metrics. In section 2 we deal briefly with the two variable warping trick. In section 3 we do the same with the warp forcing process. In section 4 we study hyperbolic forcing and prove Theorem 1. In section 5 we deal with \odot -families and prove Theorem 2.

Section 1. ϵ -close to hyperbolic metrics.

Let $\mathbb{B} = \mathbb{B}^{n-1} \subset \mathbb{R}^{n-1}$ be the unit ball, with the flat metric $\sigma_{\mathbb{R}^{n-1}}$. Write $I_\xi = (-(1+\xi), 1+\xi)$, $\xi > 0$. Our basic models are $\mathbb{T}_\xi^n = \mathbb{T}_\xi = \mathbb{B} \times I_\xi$, with hyperbolic metric $\sigma = e^{2t}\sigma_{\mathbb{R}^{n-1}} + dt^2$. In what follows we may sometimes suppress the sub index ξ , if the context is clear. The number ξ is called the *excess* of \mathbb{T}_ξ .

Let $|\cdot|$ denote the uniform C^2 -norm of \mathbb{R}^l -valued functions on $\mathbb{T}_\xi = \mathbb{B} \times I_\xi \subset \mathbb{R}^n$. Given a metric g on \mathbb{T} , $|g|$ is computed considering g as the \mathbb{R}^{n^2} -valued function $(x, t) \mapsto (g_{ij}(x, t))$ where, as usual, $g_{ij} = g(e_i, e_j)$, and the e_i 's are the canonical vectors in \mathbb{R}^n . We will say that a metric g on \mathbb{T} is ϵ -close to hyperbolic if $|g - \sigma| < \epsilon$.

A Riemannian manifold (M, g) is ϵ -close to hyperbolic if there is $\xi > 0$ such that for every $p \in M$ there is an ϵ -close to hyperbolic chart with center p , that is, there is a chart $\phi : \mathbb{T}_\xi \rightarrow M$, $\phi(0, 0) = p$, such that ϕ^*g is ϵ -close to hyperbolic. Note that all charts are defined on the same model space \mathbb{T}_ξ . The number ξ is called the *excess* of the charts (which is fixed). More generally, a subset $S \subset M$ is ϵ -close to hyperbolic if every $p \in S$ is the center of an ϵ -close to hyperbolic chart in M with fixed excess ξ .

If M has center o we say that $S \subset M$ is *radially ϵ -close to hyperbolic (with respect to o)* if, in addition, the charts ϕ respect the product structure of \mathbb{T} and $M - o = \mathbb{S}^{n-1} \times \mathbb{R}^+$, that is $\phi(\cdot, t) = (\phi_1(\cdot), t + a)$, for some constant a depending only on the ϕ . The term “radially” in the definition above refers to the decomposition of the manifold $M - o$ as a product $\mathbb{S}^{n-1} \times \mathbb{R}^+$. Note that every radially ϵ -close to hyperbolic chart ϕ can be extended to $\phi : \mathbb{B} \times (I - \{a\})$ by the same formula $\phi(x, t) = (\phi_1(x), t + a)$, but this extension may fail to be ϵ -close to hyperbolic. (Here $I - \{a\} = \{t - a, t \in I\}$.)

Let $c > 1$. A metric g on a compact manifold M is c -bounded if $|g| < c$ and $|\det g|_{C^0} > 1/c$. A set of metrics $\{g_\lambda\}$ on the compact manifold M is c -bounded if every g_λ is c -bounded.

Remarks 1.1.

1. Here the uniform C^2 -norm $|\cdot|$ is taken with respect to a fixed locally finite atlas \mathcal{A} with “extendable” charts, i.e. charts that can be extended to the (compact) closure of their domains.
2. If $\{g_t\}_{t \in I}$ is a (smooth) family and $I \subset \mathbb{R}$ is compact then, by compactness and continuity, the family $\{g_t\}$ is bounded. (Recall we are taking M compact.)
3. If $\{g_t\}_{t \in I}$ is c -bounded then clearly $\{g_{t(s)}\}_{s \in J}$ is also c -bounded, for any reindexation (or reparametrization) $t = t(s)$.

2. The Two Variable Warping Trick.

Here we review the two variable warping trick. For more details see [3]. Let h be a metric on the $(n-1)$ -sphere \mathbb{S}^{n-1} and consider the warped metric $g = \sinh^2 t h + dt^2$ on $\mathbb{S}^{n-1} \times \mathbb{R}^+$. We fix a function $\rho : \mathbb{R} \rightarrow [0, 1]$ with $\rho(t) = 0$ for $t \leq 0$ and $\rho(t) = 1$ for $t \geq 1$. Given positive numbers a and d define $\rho_{a,d}(t) = \rho(2 \frac{t-a}{d})$. Also fix an atlas $\mathcal{A}_{\mathbb{S}^n}$ on \mathbb{S}^{n-1} as before (see Remark 1 at the end of Section 1). All norms and boundedness constants will be taken with respect to this atlas.

As before let $\sigma_{\mathbb{S}^{n-1}}$ be the round metric on \mathbb{S}^{n-1} . Write

$$(2.1) \quad h_t = (1 - \rho_{a,d}(t)) \sigma_{\mathbb{S}^{n-1}} + \rho_{a,d}(t) h$$

and define the metric

$$(2.2) \quad \mathcal{T}_{a,d} g = \sinh^2 t h_t + dt^2$$

The process $g \mapsto \mathcal{T}_{a,d} g$ is the *two variable warping trick*. We have that $\mathcal{T}_{a,d} g$ satisfies property (0.3) in the Introduction.

Remark 2.3. Note that to define $\mathcal{T}_{a,d} g$ we need g to be a warped metric only for r near $a + d$.

The following is Corollary 4.5 in [3]

Theorem 2.4. *Let the metric h on \mathbb{S}^{n-1} be c -bounded. Write $g = \sinh^2(t)h + dt^2$. Then the metric $\mathcal{T}_{a,d} g$ is (B_a, ϵ) -close to hyperbolic, with charts of excess ξ , provided*

$$C \left(e^{-a} + \frac{1}{d} \right) \leq \epsilon$$

Here C is a constant depending on c , n and ξ .

For an explicit formula of C see [3] (the constant C here is denoted by C_2 in [3]). The Corollary in the Introduction [3] is obtained from 4.5 of [3] (Theorem 2.4 here) by taking $\xi = 0$.

Section 3. Spherical Cuts and Warp Forcing.

As in the Introduction, let (M^n, g) be a complete Riemannian manifold with center $o \in M$. Recall that we can write the metric on $M - \{o\} = \mathbb{S}^{n-1} \times \mathbb{R}^+$ as $g = g_r + dr^2$. We denote by $\mathbb{S}_r = \mathbb{S}_r(M) = \mathbb{S}^{n-1} \times \{r\}$ the sphere of radius r .

The metric g_r on \mathbb{S}_r is called the *warped by sinh spherical cut of g at r* , and the metric

$$\hat{g}_r = \left(\frac{1}{\sinh^2(r)} \right) g_r$$

is called the *(unwarped by sinh) spherical cut of g at r* .

Fix $r_0 > 0$. We define the warped by *sinh* metric \bar{g}_{r_0} by:

$$(3.1) \quad \bar{g}_{r_0} = \sinh^2(t) \hat{g}_{r_0} + dr^2 = \sinh^2(t) \left(\frac{1}{\sinh^2(r_0)} \right) g_{r_0} + dr^2$$

We now force the metric g to be equal to \bar{g}_{r_0} on $B_{r_0} = \mathbb{B}_{r_0}(M)$ and stay equal to g outside $B_{r_0 + \frac{1}{2}}$. For this we define the *warped forced* (on B_{r_0}) metric as:

$$(3.2) \quad \mathcal{W}_{r_0} g = (1 - \rho_{r_0}) \bar{g}_{r_0} + \rho_{r_0} g$$

where $\rho_{r_0}(t) = \rho(t - r_0)$, and $\rho : \mathbb{R} \rightarrow [0, 1]$ is smooth and such that: **(i)** $\rho|_{(-\infty, 0+\delta]} \equiv 0$, and **(ii)** $\rho|_{[1/2-\delta, \infty)} \equiv 1$, where $\delta > 0$ is small. Hence we have

$$(3.3) \quad \mathcal{W}_{r_0} g = \begin{cases} \bar{g}_{r_0} & \text{on } B_{r_0} \\ g & \text{outside } B_{r_0 + \frac{1}{2}} \end{cases}$$

Therefore the warp forced metric $\mathcal{W}_{r_0} g$ is a warped metric on B_{r_0} . We call the process $g \mapsto \mathcal{W}g$ *warp forcing*. The next result is the Main Theorem of [4]. It states that if g is ϵ -close to a hyperbolic metric then the warp forced metric $\mathcal{W}_{r_0} g$ is also close to hyperbolic.

Theorem 3.4. *Let (M, g) have center o , and $S \subset M$. If g is radially ϵ -close to hyperbolic on S with charts of excess ξ , then $\mathcal{W}_{r_0} g$ is radially η -hyperbolic on $S - B_{r_0 - (1+\xi)}$ with charts of excess $\xi - 1$, provided $\eta \geq e^{16+6\xi}(e^{-2r_0} + \epsilon)$.*

Remarks 3.5.

1. Note that warp forcing reduces the excess ξ by 1. This was one of the motivations to introduce the excess.
2. Also note that to define $\mathcal{W}_{r_0} g$ we only need g_r to be defined for $r \geq r_0$. But to apply 3.1 we need g_r to be defined for $r \geq r_0 - 1 - \xi$.
3. Notice that $\mathcal{W}_{r_0} g$ is a warped metric not just on B_{r_0} but on $B_{r_0 + \delta}$, for small $\delta > 0$.

Section 4. Hyperbolic Forcing and Proof of Theorem 1.

Let (M^n, g) have center o . As before we write $g = g_r + dr^2$. Let $r_0 > d > 0$. We define the metric $\mathcal{H}_{r_0, d}(g)$ in the following way.

First warp-force the metric g , i.e take $\mathcal{W}_{r_0} g$ (see section 3). Recall $\mathcal{W}_{r_0} g$ is a warped metric on B_{r_0} (see also remark 2.3 and remark 3 in 3.5). Hence we can use two variable warping trick (section 2) and define

$$(4.1) \quad \mathcal{H}_{r_0, d} g = \mathcal{T}_{(r_0 - d), d}(\mathcal{W}_{r_0} g)$$

The process $g \mapsto \mathcal{H}_{r_0, d}$ is called *hyperbolic forcing*. Write $h = \mathcal{H}_{r_0, d} g$. Note that h also has the form $h = h_r + dr^2$. We can explicitly describe h_r :

Proposition 4.2. *We have*

$$h_r = \begin{cases} g_r & r_0 + \frac{1}{2} \leq r \\ (1 - \rho_{r_0}(r)) \sinh^2(r) \hat{g}_{r_0} + \rho_{r_0}(r) g_r & r_0 \leq r \leq r_0 + \frac{1}{2} \\ \sinh^2(r) \left((1 - \rho_{(r_0-d),d}(r)) \sigma_{\mathbb{S}^{n-1}} + \rho_{(r_0-d),d}(r) \hat{g}_{r_0} \right) & r_0 - d \leq r \leq r_0 \\ \sinh^2(r) \sigma_{\mathbb{S}^{n-1}} & r \leq r_0 - d \end{cases}$$

where the gluing functions ρ_{r_0} and $\rho_{(r_0-d),d}$ are defined in sections 3 and 2, respectively.

Proof. We have four cases.

(i) For $r \geq r_0 + 1/2$ we have, by (3.3), that $(\mathcal{W}_{r_0}g)_r = g_r$. This together with (0.3) and (4.1) imply $h_r = g_r$.

(ii) For $r_0 \leq r \leq r_0 + 1/2$ we have, by (0.3) and (4.1), that $h_r = (\mathcal{W}_{r_0}g)_r$. And by (3.2), (3.1), we get that $(\mathcal{W}_{r_0}g)_r = (1 - \rho_{r_0}(r)) \sinh^2(r) \hat{g}_{r_0} + \rho_{r_0}(r) g_r$.

(iii) For $r_0 - d \leq r \leq r_0$ we have, by (3.1) and (3.3), that $(\mathcal{W}_{r_0}g)_r = \sinh^2(r) \hat{g}_{r_0}$. This together with (2.1) and (2.2) gives $h_r = \sinh^2(r) \left((1 - \rho_{(r_0-d),d}(r)) \sigma_{\mathbb{S}^{n-1}} + \rho_{(r_0-d),d}(r) \hat{g}_{r_0} \right)$.

(iv) For $r \leq r_0 - d$ we have, by (0.3), that $h_r = \sinh^2(r) \sigma_{\mathbb{S}^{n-1}}$. This proves the proposition.

Proposition 4.3. *The metric $h = \mathcal{H}_{r_0,d}g$ has the following properties.*

(i) *The metric h is canonically hyperbolic on B_{r_0-d} , i.e equal to $\sinh^2(r) \sigma_{\mathbb{S}^n} + dr^2$ on B_{r_0-d} .*

(ii) *We have that $g = h$ outside $B_{r_0 + \frac{1}{2}}$.*

(iii) *The metric h coincides with $\mathcal{W}_{r_0}(g_{r_0})$ outside $B_{r_0 - \frac{d}{2}}$.*

(iv) *The metric h coincides with $\mathcal{T}_{(r_0-d),d} \bar{g}_{r_0}$ on B_{r_0} .*

(v) *All the g -geodesic rays $r \mapsto ru$, $u \in \mathbb{S}^n$, emanating from the center are geodesics of (M, h) . Hence, the space (M, h) has center o . Moreover the function r (distance to the center o) is the same on the spaces (M, g) and (M, h) .*

Proof. Statements (i) and (ii) follow, for instance, from 4.2. Statement (ii) from (0.3), and (iv) from (0.3) and (3.3). Finally (v) follows from the fact that $h = h_r + dr^2$. This proves the proposition.

We are now ready to prove Theorem 1.

Proof of Theorem 1.

Recall we always take $d \geq 4(1 + \xi)$. We identify M with \mathbb{R}^n as before. Fix a finite atlas $\mathcal{A}_{\mathbb{S}^{n-1}}$ for \mathbb{S}^{n-1} (see remark 1 in 1.1). First note that $h = \sinh^2(r) \sigma_{\mathbb{S}^{n-1}} + dr^2$ (see for instance 4.3 (i)). Hence it remains to prove that h is η -close to hyperbolic outside $B_{r_0-d-1-\xi}$ (see definition of (B_a, ϵ) -close to hyperbolic metrics in the Introduction).

Let c be (a fixed constant) such that the metric \hat{g}_{r_0} is c -bounded (with respect to $\mathcal{A}_{\mathbb{S}^{n-1}}$). Let $p \in \mathbb{R}^n - B_{r_0-d-1-\xi}$ and write $p = (x, r) \in \mathbb{S}^{n-1} \times \mathbb{R}^+$. We have two cases.

First case. $r > r_0 - (1 + \xi)$.

Hence $p \notin \bar{B}_{r_0-(1+\xi)}$. By Theorem 3.4 the metric $\mathcal{W}_{r_0}g$ is radially η' -close hyperbolic outside $B_{r_0-(1+\xi)}$ with charts of excess $\xi - 1$, provided

$$\eta' \geq e^{16+6\xi} \left(e^{-2r_0} + \epsilon \right) \quad (1)$$

In particular we can find a radially η' -hyperbolic chart ϕ for $\mathcal{W}_{r_0}g$ centered at p , with excess $\xi - 1$. But by item (iii) of 4.3 and the hypothesis $4(1 + \xi) \leq d$ we get that $h = \mathcal{W}_{r_0}(g_{r_0})$ outside $B_{r_0-2(1+\xi)}$. Therefore ϕ is also a an η' -hyperbolic chart for h centered at p , with excess $\xi - 1$.

Second case. $t_0 \leq r_0 - (1 + \xi)$.

Hence $p \in \bar{B}_{r_0-(1+\xi)}$. By Theorem 2.4 (also see 3.1) the metric $\mathcal{T}_{(r_0-d),d}\bar{g}_{r_0}$ is radially η'' -hyperbolic outside $B_{r_0-d-1-\xi}$ with charts of excess ξ , provided

$$\eta'' \geq C \left(e^{-(r_0-d)} + \frac{1}{d} \right) \quad (2)$$

Here $C = C(c, n, \xi)$. In particular we can find a radially η'' -hyperbolic chart ϕ' for $\mathcal{T}_{(r_0-d),d}\bar{g}_{r_0}$ centered at p , with excess ξ . But by item (iv) 4.3 we have that $h = \mathcal{T}_{(r_0-d),d}\bar{g}_{r_0}$ on B_{r_0} . This together with the assumption $r \leq r_0 - (1 + \xi)$ imply that ϕ' is also a an η'' -hyperbolic chart for h centered at p , with excess ξ . This concludes our second case.

Hence h is radially η -close to hyperbolic with charts of excess $\xi - 1$ provided $\eta \geq \eta', \eta''$. Thus from (1) and (2) we see that can take

$$\eta \geq C \left(e^{-(r_0-d)} + \frac{1}{d} \right) + e^{16+6\xi} \left(e^{-2r_0} + \epsilon \right)$$

were $C = C(c, n, \xi)$. But $e^{-(r_0-d)} \geq e^{-2r_0}$, hence the term $e^{16+6\xi}e^{-2r_0}$ in the second summand can be absorbed into the first summand and we can take

$$\eta \geq C_1 \left(e^{-(r_0-d)} + \frac{1}{d} \right) + C_2 \epsilon$$

with $C_2 = e^{16+6\xi}$ and $C_1 = C + C_2$. This completes the proof of Theorem 1.

Section 5. Cut Limits and Limit Metrics.

Let (M^n, g) be a complete Riemannian manifold with center $o \in M$. Recall that we can write the metric on $M - \{o\} = \mathbb{R}^n - \{0\} = \mathbb{S}^{n-1} \times \mathbb{R}^+$ as $g = g_r + dr^2$, where r is the distance to o .

Fix $\xi > 0$, and let $\lambda_0 > 1 + \xi$. We say that the collection $\{g_\lambda\}_{\lambda \geq \lambda_0}$ is a \odot -family of metrics if each g_λ is a metric of the form $g_\lambda = (g_\lambda)_r + dr^2$ defined (at least) for $r > \lambda - (1 + \xi)$.

Remark. We will always assume that the family of metrics $\{g_\lambda\}$ is smooth, that is, the map $(x, \lambda) \mapsto g_\lambda(x)$ is smooth, $x \in M$, $\lambda \geq \lambda_0$

We say that the $\{g_\lambda\}$ has *cut limit at b* if there is a C^2 metric $\hat{g}_{\infty+b}$ on \mathbb{S}^{n-1} such that

$$(5.1) \quad \left| \widehat{(g_\lambda)}_{\lambda+b} - \hat{g}_{\infty+b} \right| \longrightarrow 0 \quad \text{as} \quad \lambda \rightarrow \infty$$

Remarks.

1. The metric $\widehat{(g_\lambda)}_{\lambda+b}$ is the spherical cut of g_λ at $\lambda + b$. See section 3 or (0.3).
2. The arrow above means convergence in the C^2 -norm on the space of C^2 metrics on \mathbb{S}^{n-1} . See remark 1 in 1.1.
3. The definition above implies that $(g_\lambda)_{r+b}$ is defined for large λ , even if $b < -(1 + \xi)$.
4. Note that the concept of cut limit at b depends strongly on the indexation of the family.

Let $I \subset \mathbb{R}$ be an interval (compact or noncompact). We say the \odot -family $\{g_\lambda\}$ has *cut limits on I* if the convergence in (5.1) is uniform in $b \in I$. Explicitly this means: for every $\epsilon > 0$, and $b \in I$ there are λ_* and neighborhood U of b in I such that $|\widehat{(g_\lambda)}_{\lambda+b'} - \hat{g}_{\infty+b'}| < \epsilon$, for $\lambda > \lambda_*$ and $b' \in U$. In particular $\{g_\lambda\}$ has a cut limit at b , for every $b \in I$.

Consider the \odot -family $\{g_\lambda\}$ and let $d > 0$. Apply hyperbolic forcing to get

$$h_\lambda = \mathcal{H}_{\lambda,d} g_\lambda$$

We say that the family $\{h_\lambda\}$ is the *hyperbolic forced family* corresponding to the \odot -family $\{g_\lambda\}$. Note that we can write $h_\lambda = (h_\lambda)_r + dr^2$. Using Proposition 4.1 we can explicitly describe $(h_\lambda)_r$:

$$(5.2.) \quad (h_\lambda)_r = \begin{cases} (g_\lambda)_r & \lambda + \frac{1}{2} \leq r \\ (1 - \rho_\lambda(r)) \sinh^2(r) \widehat{(g_\lambda)}_\lambda + \rho_\lambda(r) (g_\lambda)_r & \lambda \leq r \leq \lambda + \frac{1}{2} \\ \sinh^2(r) \left((1 - \rho_{(\lambda-d),d}(r)) \sigma_{\mathbb{S}^{n-1}} + \rho_{(\lambda-d),d}(r) \widehat{(g_\lambda)}_\lambda \right) & \lambda - d \leq r \leq \lambda \\ \sinh^2(r) \sigma_{\mathbb{S}^{n-1}} & r \leq \lambda - d \end{cases}$$

Proposition 5.3. *The metrics h_λ have the following properties.*

- (i) *The metrics h_λ are canonically hyperbolic on $B_{\lambda-d}$, i.e equal to $\sinh^2(r) \sigma_{\mathbb{S}^{n-1}} + dr^2$ on $B_{\lambda-d}$, provided $\lambda > d$.*
- (ii) *We have that $g_\lambda = h_\lambda$ outside $B_{\lambda+\frac{1}{2}}$.*
- (iii) *The metric h coincides with $\mathcal{W}_\lambda(g_\lambda)$ outside $B_{\lambda-\frac{d}{2}}$.*
- (iv) *The metric h coincides with $\mathcal{H}_{(\lambda-d),d}(\widehat{(g_\lambda)}_\lambda)$ on B_λ .*
- (v) *If the \odot -family $\{g_\lambda\}$ has cut limits for $b = 0$ then $\{h_\lambda\}$ has cut limits on $(-\infty, 0]$. In fact we have*

$$\hat{h}_{\infty+b} = \begin{cases} \hat{g}_\infty & b = 0 \\ (1 - \rho(2 + \frac{2b}{d})) \sigma_{\mathbb{S}^n} + \rho(2 + \frac{2b}{d}) \hat{g}_\infty & -d \leq b \leq 0 \\ \sigma_{\mathbb{S}^n} & b \leq -d \end{cases}$$

where ρ is as in section 2.

- (vi) *If we additionally assume that $\{g_\lambda\}$ has cut limits on $[0, \frac{1}{2}]$, then $\{h_\lambda\}$ has also cut limits on $[0, \frac{1}{2}]$. In fact, for $b \in [0, \frac{1}{2}]$ we have*

$$\hat{h}_{\infty+b} = (1 - \rho(b)) \hat{g}_\infty + \rho(b) \hat{g}_{\infty+b}$$

where ρ is as in section 3. Of course if $\{g_\lambda\}$ has a cut limit at $b > \frac{1}{2}$ then $\{h_\lambda\}$ has the same cut limit at b (see item 2).

- (vii) *All the rays $r \mapsto ru$, $u \in \mathbb{S}^n$, emanating from the origin are geodesics of (M, h_λ) . Hence, all spaces (M, h_λ) have center $o \in M$ and have the same geodesic rays emanating from the common center o . Moreover the function r (distance to $o \in M$) is the same on all spaces (M, h_λ) .*

Proof. Items (i), (ii), (iii) and (iv) follow from (i), (ii), (iii) and (iv) of 4.2, respectively. Also (vii) from (v) of 4.3. We prove (v). We have $\hat{h}_{\infty+b} = \lim_{\lambda \rightarrow \infty} \widehat{(g_\lambda)}_{\lambda+b}$. Hence by 5.2 (the case $\lambda - d \leq r \leq \lambda$) we get

$$(5.4) \quad \hat{h}_{\infty+b} = \lim_{\lambda \rightarrow \infty} \left(\rho_{(\lambda-d),d}(\lambda+b) \widehat{(g_\lambda)}_\lambda + (1 - \rho_{(\lambda-d),d}(\lambda+b)) \sigma_{\mathbb{S}^{n-1}} \right)$$

But from the definition of $\rho_{a,d}$ given in Section 2 we have

$$\rho_{(\lambda-d),d}(\lambda+b) = \rho\left(2\frac{(\lambda+b)-(\lambda-d)}{d}\right) = \rho\left(2 + \frac{2b}{d}\right)$$

Using this and taking the limit in (5.4) we obtain (v).

The proof of (vi) is similar to the proof of (v). Just note that in this case we use (5.2) in the case $\lambda \leq r \leq \lambda + 1/2$ and the equality $\rho_\lambda(\lambda+b) = \rho(b)$, where ρ and ρ_λ are as in Section 3. This proves the proposition.

Lemma 5.5. *If an \odot -family $\{g_\lambda\}$ has cut limits at b , then the family $\{\widehat{(g_\lambda)}_{\lambda+b}\}_\lambda$ is c -bounded, for some c .*

Proof. Since $\{g_\lambda\}$ has cut limits at 0, we have

$$(5.6) \quad \widehat{(g_\lambda)}_{\lambda+b} \rightarrow \hat{g}_{\infty+b}$$

Write $f_\lambda = \widehat{(g_\lambda)}_{\lambda+b}$. Let $c_{\infty+b}$ be such that $\hat{g}_{\infty+b}$ is $c_{\infty+b}$ -bounded. By (5.6) there is λ_* such that $|f_\lambda - g_{\infty+b}| < 1$, for every $\lambda \geq \lambda_*$. Therefore

$$|f_\lambda| \leq |f_\lambda - g_{\infty+b}| + |g_{\infty+b}| < 1 + c_{\infty+b}$$

for every $\lambda \geq \lambda_*$. Hence every f_λ is $(1 + c_{\infty+b})$ -bounded, for every $\lambda \geq \lambda_*$. On the other hand since \mathbb{S}^{n-1} and the interval $[\lambda_0, \lambda_*]$ are compact, and the map $(x, \lambda) \mapsto f_\lambda(x)$ is smooth on $\mathbb{S}^{n-1} \times [\lambda_0, \lambda_*]$, there is c_* such that every f_λ is c_* -bounded, for every $\lambda \leq \lambda_*$. Now just take $c = \max\{1 + c_{\infty+b}, c_*\}$. This proves the lemma.

Proof of Theorem 2.

By hypothesis the \odot -family $\{g_\lambda\}$ has cut limits at $b = 0$. This together with lemma 5.5 (take $b = 0$) imply that there is c such that every g_λ is c -bounded. We now apply Theorem 1 to each g_λ using the same c for all λ . This proves Theorem 2.

Finally we give a corollary of the proof of Lemma 5.5. It is an “interval version” of 5.5. The proof is similar, but we have to take track of the (now) variable b .

Corollary 5.7. *If an \odot -family $\{g_\lambda\}$ has cut limits on the compact interval I , then the family $\{\widehat{(g_\lambda)}_{\lambda+b}\}_{\lambda, b \in I}$ is c -bounded, for some c .*

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